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Infra-red dynamics of D1-branes at the conifold

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Abstract: We study the infra-red dynamics of D1-branes at the conifold. We show using methods developed to study the infra-red dynamics of $(4,4)$ theories, the infra-red degrees of freedom of the $(2,2)$ theory of a single D1-brane at the conifold is that of a linear dilaton with background charge of $\sqrt{2}$ and a compact scalar. The gauge theory of N D1-branes at the conifold is used to formulate the matrix string in the conifold background.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Conformal Field Models in String Theory.

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Contents

1. Introduction

To explore the duality between large-N gauge theories and supergravity [1, 2, 3] it is important to study cases with less supersymmetry and theories which are non conformal [4]. In this letter we study an example of such a theory. This theory is obtained on D1-branes at a conifold singularity. The conifold preserves 1/4 of the supersymmetries of the full type-IIB string theory. The theory on the D1-brane is a supersymmetric gauge theory in $1 + 1$ dimensions with 4 supercharges.

We construct the supergravity solution of this configuration. We investigate the decoupling limit and find the domains of validity of supergravity description and the super-Yang-Mills description. We see that the infra-red limit of the super-Yang-Mills corresponds to matrix string theory in the background of the conifold.

Thus the infra-red limit of the super-Yang-Mills on a single D1-brane at the conifold should correspond to world sheet of a fundamental string propagating in the background of the conifold. We study the infra-red dynamics of the D1-brane gauge theory using the methods developed for $(4, 4)$ theories by [9] following [10] and [11]. The theory on the D1-brane at the conifold has $(2, 2)$ supersymmetry. There is a 1-1 map from the moduli space of the Higgs branch of the D1-brane gauge theory to the conifold. The throat region of the Higgs branch corresponds to the singularity at the origin of the conifold. Though our theory has only $(2,2)$ supersymmetry most of the methods developed by [9] to study $(4,4)$ theories go through. Their method involves using the Coulomb variables to give an effective description of the throat region of the Higgs branch. In theories with 8 supercharges the Coulomb branch moduli space metric can receive correction only up to 1-loop. We do not have such facility for the case of $(2,2)$ theories. Nevertheless scale invariance constraints the metric to a form which enables us to extract the effective degrees of freedom. The matching of the R-symmetries in the ultra-violet and the infra-red works out just as in the case of $(4,4)$ theories.

Using these methods we are able to show that the throat region of the Higgs branch in the infra-red is captured by an $\mathcal{N}=(2,2)$ superconformal field theory consisting of a linear dilaton with background charge $Q=\sqrt{2}$ and a compact scalar. This agrees with the world sheet descriptions of strings at the conifold.

Then we consider N D1-branes at the conifold. The infra-red limit of this theory corresponds to matrix string theory in the background of the conifold. The conifold provides a background in which the ultra-violet $\mathrm{U}(1)_R$ symmetry is realized in the infra-red world sheet symmetry of the matrix string theory. We construct the leading interaction in the form of the twist operator. We note that the leading interaction is marginal.

The organization of this letter is as follows. In section 2 we study the decoupling limit of the N D1-branes at the conifold and investigate the domains of validity of the supergravity and the gauge theory. Section 3 analyses the infra-red dynamics of the gauge theory of a single D1-brane at the conifold. Section 4 formulates matrix string theory in the background of the conifold. We conclude in section 5. The appendix contains details of the supergravity solution.

2. Supergravity and the large-N limit of the D1-brane theory at the conifold

In this section we study the supergravity solution of N D1-branes at a conifold singularity in the decoupling limit. We consider the configuration in which the D1-branes are aligned along the x^1 co-ordinate. The supergravity solution is given by (The verification of this solution is given in the appendix.)

$$\begin{aligned} ds^2 = {} & f^{-1/2}\left(-dx_0^2+dx_1^2\right)+ \\ & + f^{1/2}\Big[dr_1^2+r_1^2\,d\chi^2+dr_2^2+\frac{r_2^2}{9}\left(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2\right)^2+ \\ & + \frac{r_2^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2+\sin^2\theta_i\,d\phi_i^2\right)\Big]\,, \end{aligned}$$

$$e^{(\Phi-\Phi_\infty)}=f^{1/2}\,,\qquad B_{01}=-\frac{1}{2}\left(f^{-1}-1\right),\qquad f=1+\frac{NCg_s\alpha'^3}{(r_1^2+r_2^2)^3}\,. \tag{2.1}$$

The co-ordinates transverse to the D1-brane are $r_1, \chi, r_2, \psi, \theta_i, \phi_i$. i runs from 1 to 2. r_1, χ are polar co-ordinates of R^2. The remaining coordinates parameterize the conifold. The angular part of the conifold is parametrized by ψ, θ_i, ϕ_i, the radial part is parametrized by r_2. (θ_1, ϕ_1) and (θ_2, ϕ_2) parameterizes $S^2 \times S^2$ as polar co-ordinates. $\psi \in [0, 4\pi]$ parameterizes the U(1) fiber over $S^2 \times S^2$. C is fixed by charge quantization. It is given by $C = 864\pi^2/(16 + 15\pi)$. This configuration preserves 4 supersymmetries out of the 32 supersymmetries of type-IIB supergravity.

We study the decoupling limit of this system as in [4]. It takes the form

$$U_1 = \frac{r_1}{\alpha'} = \text{fixed}, \qquad U_2 = \frac{r_2}{\alpha'} = \text{fixed}, \qquad g_{\rm YM}^2 = \frac{1}{2\pi}\frac{g_s}{\alpha'} = \text{fixed}, \quad \alpha' \to 0\,. \quad (2.2)$$

The metric and the dilaton of the supergravity solution in this limit is given by

$$\begin{aligned} \frac{ds^2}{\alpha'} &= \frac{U^3}{g_{\rm YM}\sqrt{2\pi NC}}\left(-dx_0^2 + dx_1^2\right) + \\ &\quad + \frac{g_{\rm YM}\sqrt{2\pi NC}}{U^3}\Bigg[dU_1^2 + U_1^2 d\chi^2 + dU_2^2 + \frac{U_2^2}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 + \\ &\qquad\qquad + \frac{U_2^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2\right)\Bigg]\,, \\ e^{\Phi} &= \sqrt{\frac{8\pi^3 NC g_{\rm YM}^6}{U^6}}\,, \end{aligned} \quad (2.3)$$

where $U^2 = U_1^2 + U_2^2$.

Let us now discuss the domains of validity of various descriptions of this system. We will use the co-ordinate U to set the energy scale at which we wish to look at the system. We will start with the high energies. For $U \gg g_{\rm YM}\sqrt{N}$ the super-Yang-Mills perturbation theory can be trusted.

To find out when the supergravity solution given in (2.3) can be trusted let us estimate the curvature of the solution. Using the equation of motion an estimate of the curvature in string units is given by

$$\begin{aligned} \alpha' R &\sim g^{U_1U_1}\partial_{U_1}\Phi\,\partial_{U_1}\Phi + g^{U_2U_2}\partial_{U_2}\Phi\partial_{U_2}\Phi \\ &\sim \frac{U}{g_{\rm YM}\sqrt{N}}\,. \end{aligned} \quad (2.4)$$

To trust supergravity the curvature should be small. Furthermore we need to ensure that the expansion in string coupling is valid. This is requires e^{Φ} in (2.3) to be small. Thus the supergravity solution is valid for $g_{\rm YM}N^{1/6} \ll U \ll g_{\rm YM}\sqrt{N}$. In addition to this condition, we must have $U \ll U_2^{2/3}(Ng_{\rm YM}^2)^{1/6}$. The latter condition arises from the fact that there is a curvature singularity at $U_2 = 0$.[1]

[1]This point was raised by G. Horowitz and N. Itzhaki.

In the region $U \ll g_{\rm YM}N^{1/6}$ we can use S-duality to study the solution. Performing S-duality on the near horizon solution in (2.3) we obtain

$$
\begin{aligned}
\frac{ds^2}{\tilde{\alpha}'} &= \frac{U^6}{g_{\rm YM}^4 4\pi^2 NC}\left(-dx_0^2+dx_1^2\right)+ \\
&\quad+\frac{1}{2\pi g_{\rm YM}^2}\Bigg[dU_1^2+U_1^2 d\chi^2+dU_2^2+\frac{U_2^2}{9}\left(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2\right)^2+ \\
&\qquad\qquad+\frac{U_2^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2+\sin^2\theta_i d\phi_i^2\right)\Bigg], \\
e^{-\Phi} &= \sqrt{\frac{8\pi^3 NC g_{\rm YM}^6}{U^6}}, \qquad (2.5)
\end{aligned}
$$

where $\tilde{\alpha}' = g_s\alpha'$. The supergravity solution in (2.5) is the near horizon geometry of N fundamental strings at the conifold. An estimate of the curvature in string units can be performed as before, this gives the following

$$
\tilde{\alpha}' R \sim \frac{g_{\rm YM}^2}{U^2}\,. \qquad (2.6)
$$

This shows that this S-dual supergravity description is valid for $U \gg g_{\rm YM}$. As before we also have an additional condition $\alpha' U_2 \gg 1$ so as to avoid the curvature singularity at $U_2 = 0$. For small U we have no supergravity description regardless of N.

If one is able to capture the infra-red behaviour of the super-Yang-Mills on the D1-branes then it is clear that one obtains a non-perturbative description of propagation of fundamental strings in the background of the conifold. This is analogous to case of D1-branes in flat space. The infra-red behaviour of super-Yang-Mills with gauge group U(N) and 16 supercharges provides a non-perturbative description of string theory [5, 6, 7]. It is interesting to compare this also with the infra-red behaviour of the D1/D5 system. There the infra-red dynamics of the D1-branes captures the DLCQ of the little string theories in the Higgs branch. The Coulomb branch conformal field theory gives a non-perturbative description of strings in the background of Neveu-Schwarz 5-branes.

3. Infra-red dynamics of a single D1-brane

In this section we show that the gauge theory of a single D1-brane at the conifold in the infra-red flows to a superconformal field theory of a string in the background of the conifold.

3.1 The gauge theory

The gauge theory on the D1-brane at the conifold consists of a $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge theory with $(2,2)$ supersymmetry in $1+1$ dimensions [8]. The matter content of this theory consists of two sets of chiral multiplet, A_i, and B_i with $i = 1, 2$. The A's and B's are charged as $(1,-1)$ and $(-1,1)$, respectively. The diagonal U(1) decouples. It corresponds to the free U(1) gauge multiplet on the single D1-brane. The 2 scalars of this gauge multiplet represent motion of the D1-brane along r_1 and χ. Under the relative U(1) the A's have charge +1 and the B's have charge −1. The D-term of the U(1) vector multiplet is given by

$$D = |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 \,. \qquad (3.1)$$

We consider the case in which both the Fayet-Iliopoulos term and the theta term in the lagrangian are set to zero. The conifold is realized as the moduli space of vacuum of the Higgs branch of this theory. Setting the D-term to zero and dividing by the gauge group U(1) realizes the conifold. The complex coordinates of the conifold are given by

$$z_1 = A_1 B_1, \qquad z_2 = A_2 B_2, \qquad z_3 = A_1 B_2, \qquad z_4 = A_2 B_1 \qquad (3.2)$$

with

$$z_1 z_2 - z_3 z_4 = 0 \,. \qquad (3.3)$$

Therefore the infrared theory is a superconformal field theory with the conifold as its target space. The central charge in the Higgs branch is given by counting the gauge invariant degrees of freedom. This is seen to be 9. To isolate the description of the conifold at the singularity we describe the conifold as follows.

If point (a_1, a_2, a_3, a_4) satisfies (3.3), and $a_i \neq 0$ then one can obtain another solution which is given by $\sigma^{1/2}(a_1, a_2, a_3, a_4)$. Here σ is a complex number. This particular scaling is chosen so that the complex polynomial describing the conifold in (3.3) is homogeneous of degree 1. Therefore the conifold can be described by the space

$$\sigma \times (z_1 z_2 - z_3 z_4 = 0)/\sigma \,, \qquad (3.4)$$

where the space $(z_1 z_2 - z_3 z_4)/\sigma$ is 2 complex-dimensional hypersurface $z_1 z_2 - z_3 z_4 = 0$ in the 3 complex-dimensional weighted projective space $\mathbb{WCP}^3_{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}$.

The central charge of the $\mathcal{N} = (2, 2)$ superconformal field theory on the hypersurface in the weighted projective space is zero. Thus it does not contain any degrees of freedom. The entire central charge 9 of the superconformal field theory of the conifold thus resides on the superconformal field theory on the one-dimensional complex space parametrized by σ. It is clear this space is endowed with a non-trivial metric. To obtain this metric we examine the theory at the origin of the Higgs branch. At the origin of the Higgs branch there is a "singularity" as the Fayet-Iliopoulos term and the theta term are set to zero. This corresponds to the $z_i = 0$ point on the conifold. In the infra-red we can describe the throat region of the Higgs branch using the Coulomb variables. This is because in the infra-red, the vector multiplet is an auxiliary field. The kinetic terms of the vector multiplet decouples as they are irrelevant. Then the Higgs fields can be written in terms of the vector multiplet using equations of motion. This method of describing the Higgs branch using the Coulomb variables was done in [9] of $N=(4,4)$ gauge theories. We obtain the metric on the complex line parametrized by σ by appealing to the description of the Higgs branch in terms of the Coulomb branch.

To proceed with the analysis we write down the lagrangian for the relative U(1). We follow the convention of [12] but work with euclidean world sheet metric. We set $y^0=-iy^2$ in the formulae of [12].

$$
\begin{aligned}
L &= L_{\text{matter}} + L_{\text{gauge}}\,,\\
L_{\text{matter}} &= \int d^2y\, D_\mu \bar{A}_i D^\mu A_i + D_\mu \bar{B}_i D^\mu B_i + 2\sigma\bar{\sigma}\bar{A}_i A_i + 2\sigma\bar{\sigma}\bar{B}_i B_i -\\
&\quad - i\bar{\psi}^A_{-i}(D_1+iD_2)\psi^A_{-i} - i\bar{\psi}^B_{-i}(D_1+iD_2)\psi^B_{-i} -\\
&\quad - i\bar{\psi}^A_{+i}(D_1-iD_2)\psi^A_{+i} - i\bar{\psi}^B_{+i}(D_1-iD_2)\psi^B_{+i} +\\
&\quad + \sqrt{2}\left(\bar{\sigma}\bar{\psi}^A_{+i}\,\psi^A_{-i} + \sigma\bar{\psi}^A_{-i}\,\psi^A_{+i}\right) - \sqrt{2}\left(\bar{\sigma}\bar{\psi}^B_{+i}\,\psi^B_{-i} + \sigma\bar{\psi}^B_{-i}\,\psi^B_{+i}\right) +\\
&\quad + i\sqrt{2}\bar{A}_i\left(\psi^A_{-i}\,\lambda_+ - \psi^A_{+i}\,\lambda_-\right) + i\sqrt{2}A_i\left(\bar{\lambda}_-\,\bar{\psi}^A_{+i} - \bar{\lambda}_+\,\bar{\psi}^A_{-i}\right) +\\
&\quad + i\sqrt{2}\bar{B}_i\left(\psi^B_{-i}\,\lambda_+ - \psi^B_{+i}\,\lambda_-\right) + i\sqrt{2}B_i\left(\bar{\lambda}_-\,\bar{\psi}^B_{+i} - \bar{\lambda}_+\,\bar{\psi}^B_{-i}\right) -\\
&\quad - D\bar{A}_i A_i + D\bar{B}_i B_i\,,\\
L_{\text{gauge}} &= \frac{1}{g^2_{\text{YM}}}\int d^2y \left(\frac{1}{2}F^2_{01} - \frac{1}{2}D^2 - i\bar{\lambda}_+(\partial_1 - i\partial_2)\lambda_+ -\right.\\
&\qquad\qquad \left. - i\bar{\lambda}_-(\partial_1 + i\partial_2)\lambda_- + \partial_\mu\bar{\sigma}\partial^\mu\sigma\right). \qquad (3.5)
\end{aligned}
$$

The superpotential on a single D1-brane is zero, therefore we have set the F-terms to zero. The vector multiplet consists of fields $F_{01}, \sigma, \lambda_+, \lambda_-$. σ is a complex scalar corresponding to the components of a 4-dimensional gauge field along 2 compact directions. The gauginos λ_+, λ_- are complex Weyl fermions. The super partners of the chiral multiplet A_i, B_i are $\psi^A_{i+,-}, \psi^A_{i+,-}$, respectively. They are also complex Weyl fermions in 2 dimensions.

The scaling dimension of the Yang-Mills coupling is 1. Therefore in the infra-red the coupling tends to ∞. This ensures that the kinetic term for the vector multiplet L_{gauge} decouples in the infra-red. In the Higgs branch the scaling dimension of the scalars A_i, B_i is zero, and its superpartners have dimension 1/2. The scalars σ and the gauge boson have scaling dimension 1. Its superpartners have scaling dimension 3/2. This is more evidence that the operators in L_{gauge} are irrelevant. Therefore in the infra-red, the lagrangian is restricted to only L_{matter}.

We now can integrate over the auxiliary vector multiplet in L_{matter}. This forces the D-term to be set to zero and one obtains the Higgs branch as a $\mathcal{N}=(2,2)$ superconformal field theory over the conifold. However in order to describe the theory near the singularity we will follow the method of [9]. Here the vector multiplets are regarded as composite operators on the Higgs branch. They are roughly given by

$$\sigma = \frac{1}{\sqrt{2}} \frac{\bar{\psi}^B_{-i}\, \psi^B_{+i} - \bar{\psi}^A_{-i}\, \psi^A_{+i}}{\bar{A}_i A_i + \bar{B}_i B} \, . \tag{3.6}$$

This amounts to integrating out the chiral multiplets in L_{matter}. This was argued in [9] to be valid at large values of σ. From (3.6) we see that is valid roughly for small values of the chiral multiplets A_i, B_i. Thus by large values of σ we are probing the singularities of the Higgs branch. We will discuss the systematics of the expansion as we perform the 1-loop computation.

3.2 The one-loop calculation

The terms in the lagrangian which are relevant for the 1-loop calculation are

$$\begin{aligned} L = \int d^2y \partial_\mu \bar{A}_i\, \partial^\mu A_i &+ \partial_\mu \bar{B}_i\, \partial^\mu B_i + 2\sigma\bar{\sigma}\bar{A}_i A_i + 2\sigma\bar{\sigma}\bar{B}_i B_i - \\ &- i\bar{\psi}^A_{-i}(\partial_1 + i\partial_2)\psi^A_{-i} - i\bar{\psi}^B_{-i}(\partial_1 + i\partial_2)\psi^B_{-i} - \\ &- i\bar{\psi}^A_{+i}(\partial_1 - i\partial_2)\psi^A_{+i} - i\bar{\psi}^B_{+i}(\partial_1 - i\partial_2)\psi^B_{+i} + \\ &+ \sqrt{2}\left(\bar{\sigma}\, \bar{\psi}^A_{+i}\, \psi^A_{-i} + \sigma\bar{\psi}^A_{-i}\, \psi^A_{+i}\right) - \sqrt{2}\left(\bar{\sigma}\bar{\psi}^B_{+i}\, \psi^B_{-i} + \sigma\bar{\psi}^B_{-i}\, \psi^B_{+i}\right) \, . \end{aligned} \tag{3.7}$$

Integrating out the chiral multiplets to 1-loop gives the following terms in the action

$$\begin{aligned} S_{\text{1-loop}} = & -4\ln\left[\det(-\Box + 2\bar{\sigma}\sigma)\right] + 2\ln\left[\det\begin{pmatrix} -i(\partial_1 + i\partial_2) & +\sqrt{2}\sigma \\ +\sqrt{2}\bar{\sigma} & i(\partial_1 - i\partial_2) \end{pmatrix}\right] + \\ & + 2\ln\left[\det\begin{pmatrix} -i(\partial_1 + i\partial_2) & -\sqrt{2}\sigma \\ -\sqrt{2}\bar{\sigma} & i(\partial_1 - i\partial_2) \end{pmatrix}\right] . \end{aligned} \tag{3.8}$$

On simplification one gets

$$\begin{aligned} S_{\text{1-loop}} = & \operatorname{Tr}\ln\left[1 + \begin{pmatrix} 0 & \frac{1}{-\Box + 2\sigma\bar{\sigma}}[-\sqrt{2}(\partial_1 + i\partial_2)]\sigma \\ \frac{1}{-\Box + 2\sigma\bar{\sigma}}[\sqrt{2}(\partial_1 - i\partial_2)]\bar{\sigma} & 0 \end{pmatrix}\right] + \\ & + \operatorname{Tr}\ln\left[1 + \begin{pmatrix} 0 & \frac{1}{-\Box + 2\sigma\bar{\sigma}}[\sqrt{2}(\partial_1 + i\partial_2)]\sigma \\ -\frac{1}{-\Box + 2\sigma\bar{\sigma}}[i\sqrt{2}(\partial_1 - i\partial_2)]\bar{\sigma} & 0 \end{pmatrix}\right] . \end{aligned} \tag{3.9}$$

From (3.9) it is clear that the expansion parameter is $d\sigma/\sigma^2$. We expect this expansion to be valid for $d\sigma \ll \sigma^2$. Thus, we obtain a good description of the Higgs branch in terms of the Coulomb variables for large σ, which according to (3.6) corresponds to regions near the singularity. After further simplification the leading order in the velocity expansion is given by

$$\begin{aligned} S_{\text{1-loop}} &= 4\int d^2y(\partial_1 + i\partial_2)\sigma(\partial_1 - i\partial_2)\bar{\sigma}\, \operatorname{Tr}\left[\frac{1}{(-\Box + 2\sigma\bar{\sigma})^2}\right] \\ &= 4\int d^2y(\partial_1 + i\partial_2)\sigma(\partial_1 - i\partial_2)\bar{\sigma} \int \frac{d^2k}{4\pi^2} \frac{1}{(k^2 + 2\sigma\bar{\sigma})^2} \, , \end{aligned} \tag{3.10}$$

σ is a complex scalar which represents the 2 coordinates of the Coulomb branch. Writing these in polar coordinates and performing the integration we obtain the following metric on the moduli space.

$$ds^2 = \frac{dr^2}{2\pi r^2} + \frac{d\theta^2}{2\pi}\,. \tag{3.11}$$

There is also a torsion given by

$$B_{r\theta} = \frac{1}{2\pi r}\,. \tag{3.12}$$

The torsion is a pure gauge term. The space is topologically $R \times S^1$. This does not have any non-trivial closed 2-cycles. Thus there is no obstacle for gauging away the torsion. The gauge transformation is given by

$$B_{r\theta} = \partial_r \Lambda_\theta - \partial_\theta \Lambda_r\,. \tag{3.13}$$

Setting $\Lambda_r = 0$ gives $\Lambda_\theta = \ln r/2\pi$. We have performed only a 1-loop calculation for the moduli space metric. The moduli space metric in (3.11) can be argued to be of the form given by the 1-loop result by scale invariance. σ is a scalar with dimension 1 in the ultra-violet gauge theory. The only scale invariant term one can write for the moduli space metric is $d\sigma d\bar{\sigma}/\sigma\bar{\sigma}$. We have determined the coefficient by the 1-loop calculation. A similar 1-loop calculation was done for the $(4,4)$ relevant for the D1/D5 system in [13]. In $(4,4)$ gauge theories a non-renormalization theorem determines the moduli space metric by the 1-loop result [14, 13, 15]. For the $(2,2)$ case there is no such theorem, but conformal invariance constraints the metric up to a numerical coefficient. To determine the infra-red degrees of freedom, we do not need this coefficient.

3.3 The infra-red degrees of freedom

It is clear from the moduli space metric in (3.11) that in the infra-red the dimension of the field r is not determined. We define a scalar field ϕ as $r = e^{-\phi/2}$. Then the kinetic term for ϕ is just that of a free field. The field ϕ can behave like a linear dilaton. The dimension of r is specified only if one knows the background charge of the linear dilaton ϕ. We determine the background charge by requiring that the central charge of the infra-red $\mathcal{N} = (2,2)$ superconformal field theory to be 9 which as we saw before was the central charge of the Higgs branch. We will justify this using R-symmetries in section 3.4.

The bosonic fields capturing the infra-red dynamics are the linear dilaton ϕ, the compact scalar θ. The radius of the compact scalar is 2. This is the value determined by the 1-loop calculation. The bosonic part of the action is given by

$$L = \frac{1}{8\pi} \int d^2y \sqrt{g} (g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi - QR\phi + g^{\mu\nu} \partial_\mu \theta\, \partial_\nu \theta)\,, \tag{3.14}$$

where R is the world sheet curvature and Q is the background charge of the linear dilaton. We have redefined θ so that we can use the $\alpha' = 2$ convention. The superpartners of ϕ and θ are free fermions as the curvature of the moduli space $R_{r\theta r\theta}$ is zero.

Now we can evaluate the central charge. The total central charge of the infra-red superconformal field theory is

$$c = \frac{3}{2} + 3Q^2 + \frac{3}{2} \,. \tag{3.15}$$

Demanding that the central charge be 9 gives $Q^2 = 2$. The sign of the background charge is determined by requiring that the singularity is at the strong coupling region.This fixes the background charge to be $Q = \sqrt{2}$. At this point we mention that in the $(4,4)$ case the sign was determined by the fact that the conformal field theory of the Higgs branch and the conformal field theory of the Coulomb branch could be considered as two different subalgebras of the large-$\mathcal{N} = (4,4)$ superconformal algebra.

We note that demanding that the central charge be 9 does not determine the infra-red conformal field theory completely. The central charge of a conformal field theory is unchanged if it is deformed by a marginal operator. One obvious marginal deformation is the radius of the compact scalar. Thus we cannot determine the radius of the compact scalar in the infra-red limit. We determined the radius only using a 1-loop calculation. This certainly can change in the infra-red limit.

To be explicit we write down the holomorphic generators of the infra-red $\mathcal{N} = (2,2)$ superconformal field theory.

$$\begin{aligned}
\bar{G} &= \psi \partial_z \bar{X} + \partial_z \psi \,, \\
G &= \bar{\psi} \partial_z X + \partial_z \bar{\psi} \,, \\
J_R &= -\psi\bar{\psi} + i\sqrt{2}\, \partial_z X^2 \,, \\
T &= -\partial_z \bar{X} \partial_z X - \frac{1}{\sqrt{2}} \partial_z^2 X^1 + \frac{1}{2} \left(-\partial_z \psi \bar{\psi} + \psi \partial_z \bar{\psi} \right) ,
\end{aligned} \tag{3.16}$$

where

$$X = \frac{X^1 + iX^2}{\sqrt{2}}, \qquad \bar{X} = \frac{X^1 - iX^2}{\sqrt{2}}, \qquad \psi = \frac{\psi^1 + i\psi^2}{\sqrt{2}} \qquad \bar{\psi} = \frac{\psi^1 - i\psi^2}{\sqrt{2}}. \tag{3.17}$$

The field X^1 corresponds to the linear dilaton ϕ and X^2 corresponds to the compact scalar θ. The fermions ψ^1, ψ^2 are the superpartners of X^1 and X^2, respectively. There is a similar set of anti-holomorphic generators.

3.4 Comparison of the R-symmetries

We now compare the R-symmetries of the ultra-violet and the infra-red theory and show that our identification of the infra-red degrees of freedom is justified.

In the D1-brane gauge theory there is a $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ R-symmetry. The fields and their charges under $\mathrm{U}(1)_L$ are as follows [12], $(\psi_{+i}^A, \psi_{+i}^B, e^{i\theta}, \lambda_-)$ have charges

$(-1,-1,1,1)$. The rest of the fields are uncharged under $\mathrm{U}(1)_L$. The fields $(\psi^A_{-i}, \psi^B_{-i}, e^{i\theta}, \lambda_+)$ have charges $(-1,-1,-1,1)$ under $\mathrm{U}(1)_R$. The rest of the fields are uncharged under $\mathrm{U}(1)_R$. The infra-red behaviour of state localized far along the Higgs branch is approximately free. In an $\mathcal{N}=(2,2)$ free super conformal field theory with a flat metric the R-symmetry does not act on the bosons. The $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ does not act on the bosons of the chiral multiplet A_i and B_i. Thus it is natural to identify the R-symmetry of the D1-brane gauge theory with that of the $\mathcal{N}=(2,2)$ superconformal field theory of the Higgs branch. As the $\mathcal{N}=(2,2)$ superconformal algebra relates the R-symmetry to the central charge we are justified in requiring that the central charge of the infra-red superconformal field theory of the Higgs branch be 9.

The Coulomb branch is parametrized by the bosons $|\sigma|, e^{i\theta}$. As $e^{i\theta}$ is charged under $\mathrm{U}(1)_L\times\mathrm{U}(1)_R$ this cannot be the R-symmetry of the conformal field theory of the Coulomb branch. It must be as the theory flows to the infra-red on the Coulomb branch an R-symmetry is developed. This is similar to the case of $(4,4)$ gauge theories [10]. In these theories the $\mathrm{SU}(2)_R$ symmetry of the gauge theory in the ultra-violet is a candidate for the R-symmetry of the conformal field theory of the Coulomb branch. This symmetry is enhanced to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ as the theory flows to the infra-red in the Coulomb branch.

Let us now examine from the infra-red degrees of freedom of the Higgs branch in the throat region whether the R-symmetry acts similar to $\mathrm{U}(1)_L\times\mathrm{U}(1)_R$. Let us focus on the holomorphic part, the anti-holomorphic part follows similarly. The field $(e^{iX^2/\sqrt{2}}, \psi)$ are charged as $(1,1)$. This is what is expected under the identification of $\mathrm{U}(1)_L\times\mathrm{U}(1)_R$ as the R-symmetry of the conformal field theory of the Higgs branch.

3.5 Fundamental strings at the conifold

We have seen in section 2 that using arguments of [4] that the infra-red theory of the D1-brane at the conifold should correspond to that of the world sheet of fundamental strings in the background of the conifold. We wish to compare the infra-red theory obtained with what is known about string propagation at the conifold.

String propagation at singularities have been studied recently in a series of works [16, 17, 18]. It is seen from these works that string propagation at the conifold is described by a linear dilaton theory with background charge $Q=\sqrt{2}$ and a compact scalar. The effective degrees of freedom of the infra-red D1-brane gauge theory precisely matches with this.

String propagation at the resolved conifold $z_1^2+z_2^2+z_3^2+z_4^2=\mu$, has been discussed in [19, 20, 21]. In these series of works it was argued that the world sheet theory was given by an $\mathcal{N}=(2,2)$ $\mathrm{SL}(2,R)/\mathrm{U}(1)$ Kazama-Suzuki model at level 3. The $\mathrm{SL}(2,R)/\mathrm{U}(1)$ model away from the origin consists of a linear dilaton with $Q=\sqrt{2}$ and a compact scalar at the self dual radius [22, 23]. This also agrees with the infra-red degrees of freedom of the D1-brane gauge theory.

4. Matrix String theory in a conifold background

In this section we use the lagrangian of N D1-branes in a conifold background to formulate matrix string theory in this background. To formulate matrix string theory we need that the spatial coordinate of the two-dimensional Yang-Mills to be compact [5, 6, 7]. To the knowledge of the author, matrix string theory has not been formulated in a background with 8 supersymmetries.

The lagrangian of N D1-branes at a conifold is that constructed in [8] dimensionally reduced to two dimensions. It consists of a $\mathrm{U}(N)\times\mathrm{U}(N)'$ gauge theory with $(2,2)$ supersymmetry. We will use $d=4$ $N=1$ supersymmetry nomenclature to classify our fields. There are two gauge multiplets corresponding to the two gauge groups. The bosonic fields of the gauge multiplet consist of two bosons transforming in the adjoint of the corresponding gauge group. The fermions of the gauge multiplet are complex Weyl fermions in two dimensions. They are spinors of SO(2) the symmetry in the transverse directions parametrized by r_1 and χ. This can be seen from the fact that they arise from dimensional reduction of complex Weyl fermions of 4 dimension. They transform in the adjoint representation of the gauge group. We list the fields of the gauge multiplets below.

$$\begin{array}{ll} \text{Bosons} & A_\mu, A'_\mu, X_a, X'_a\,, \\ \text{Fermions} & \lambda_+, \lambda_-, \lambda'_+\lambda'_-\,, \end{array} \tag{4.1}$$

where $a=1,2$. The primes over the field variables indicate that they transform under the gauge group $\mathrm{U}(N)'$. There are 4 chiral multiplets arranged in two sets A_i and B_i, $i=1,2$. The A_i transform as $\mathrm{U}(N)\times\overline{\mathrm{U}(N)'}$, while the B_i transform as $\overline{\mathrm{U}(N)}\times\mathrm{U}(N)'$. We use the capital A's and B's to indicate the superfields as well as the bosonic component. The fields of the chiral multiplets are

$$\begin{array}{ll} \text{Superfield } A_i & A_i, \psi^A_{-i}\, \psi^A_{+i}\,, \\ \text{Superfield } B_i & B_i, \psi^B_{-i}\, \psi^B_{+i}\,. \end{array} \tag{4.2}$$

The lagrangian has a superpotential given by

$$W = \frac{1}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,A_iB_kA_jB_l\,. \tag{4.3}$$

We choose the gauge coupling of the two gauge groups to be identical. The bosonic potential is given by

$$\begin{aligned} U = {} & g^2_{\mathrm{YM}}\sum_i\left|\frac{\partial W}{\partial A_i}\right|^2 + g^2_{\mathrm{YM}}\sum_i\left|\frac{\partial W}{\partial B_i}\right|^2 + \frac{1}{2g^2_{\mathrm{YM}}}\,\mathrm{Tr}\,D^2 + \frac{1}{2g^2_{\mathrm{YM}}}\,\mathrm{Tr}\,D'^2 + \\ & + A^*_i\left(X_1^2+X_2^2\right)A_i + A^*_i\left(X_1'^2+X_2'^2\right)A_i + B^*_i\left(X_1^2+X_2^2\right)B_i + \\ & + B^*_i\left(X_1'^2+X_2'^2\right)B_i + \frac{1}{2\tilde g^2_{\mathrm{YM}}}[X_1,X_2]^2 + \frac{1}{2g^2_{\mathrm{YM}}}[X'_1,X'_2]^2\,. \end{aligned} \tag{4.4}$$

Perturbative type-IIA string theory is realized out of the matrix description in the $g_{\rm YM} \to \infty$. From the superpotential we see that this limit selects out a vacuum. One such vacuum is in which all the A's and B's are diagonal. We analyze the theory around this vacuum. The gauge group is broken down to $\mathrm{U}(1)^N$ in this vacuum. Each of the $\mathrm{U}(1)$ corresponds to the center of mass $\mathrm{U}(1)$ for the single D1-brane considered in section 3. The Weyl group of $\mathrm{U}(N) \times \mathrm{U}(N)'$ acts on the vacuum as

$$A \longrightarrow SAS'^{\dagger}\,, \qquad B \longrightarrow S'BS^{\dagger}\,. \tag{4.5}$$

We see that the Weyl group of $\mathrm{U}(N) \times \mathrm{U}(N)'$ transform A and B to values which are diagonal with the entries permuted only if S and S' are the same element of the Weyl group. Such transformations takes one vacuum to another. The gauge invariant vacuum is given by identifying these. The D terms for the relative $\mathrm{U}(1)$ for each of the D1-brane reduce to

$$|A_1^m|^2 + |A_2^m|^2 - |B_1^m|^2 - |B_2^m|^2 = 0\,, \tag{4.6}$$

where $m = 1, \ldots, N$. N copies of the conifold is realized as the moduli space of vacuum. The complex coordinates of the N copies of the conifold are given by

$$z_1^m = A_1^m B_1^m, \qquad z_2^m = A_2^m B_2^m, \qquad z_3^m = A_1^m B_2^m, \qquad z_4 = A_2^m B_1^m\,. \tag{4.7}$$

Thus the conformal field theory that describes the infra-red limit of this gauge theory is a sigma model on the orbifold target space

$$\frac{(R^2 \times C)^N}{\mathrm{S}(N)}\,, \tag{4.8}$$

where $\mathrm{S}(N)$ is the symmetric group and C stands for the conifold. The R^2 refers to the transverse directions parametrized by r_1 and χ. These arise from the values of scalars X_a^m corresponding to the diagonal $\mathrm{U}(1)$'s. The question of finding backgrounds for matrix string theory where a $\mathrm{U}(1)_R$ is present in the ultraviolet gauge theory which appears as the world sheet $\mathrm{U}(1)_R$ was raised recently in [24]. In the ultraviolet of this matrix theory, there is a superpotential. Therefore the chiral multiplets are charged under the $\mathrm{U}(1)_R$ in the ultraviolet. Thus this R-symmetry cannot be the $\mathrm{U}(1)_R$ of the world sheet theory in the infrared.[2]

Using the results of section 3, the conformal field theory on the conifold near the singularity that captures the infra-red limit of the gauge theory is given by the orbifold

$$\frac{(R^2 \times R_\phi \times S^1)^N}{\mathrm{S}(N)}\,, \tag{4.9}$$

[2]The author thanks E. Silverstein and Y.S. Song for pointing out that for this matrix theory too, the question in [24] is unresolved, correcting the erroneous conclusion in the earlier draft.

where R_ϕ stands for the linear dilaton with background charge $Q = \sqrt{2}$ and S^1 refers to the compact scalar. At this point let us examine the domain of validity of the super conformal field theory on the orbifold (4.9). The super conformal field theory on the orbifold (4.9) is valid when $g_{\rm YM} \to \infty$ and $z \to 0$. Here z stands for all the co-ordinates in (4.7). Now, it is important that there exists a domain in these limits that the mass of the off-diagonal chiral multiplets can be neglected. The mass of the off-diagonal chiral multiplets roughly goes as $g_{\rm YM}^2 z^2$. Thus, the super conformal field theory on the orbifold is valid in the limits $g_{\rm YM} \to \infty$, $z \to 0$ and $g_{\rm YM}^2 z^2 \to \infty$.

We would like to construct the leading interaction vertex represented by the twist operator corresponding to the $\mathbb{Z}_2$ conjugacy class of the permutation group. For this we focus on the orbifold

$$\frac{(R^2 \times R_\phi \times S^1)^2}{\mathbb{Z}_2}. \tag{4.10}$$

Going over to center of mass and relative coordinate, conformal field theory on the following target space is realized

$$\left(R^2 \times R_\phi \times S^1\right) \times \frac{(R^3 \times S^1)}{\mathbb{Z}_2}. \tag{4.11}$$

Here the linear dilaton R_ϕ has background charge $Q = \sqrt{2} \times \sqrt{2}$. From the fact that the orbifold is a $(R^3 \times S^1)/\mathbb{Z}_2$, the interaction vertex is represented by the twist operator that is marginal. This is unlike the case of D1-branes in flat space where the leading interaction was irrelevant [7]. The fact that there is a marginal operator in the infrared in this case is puzzling. This results perhaps from the fact that the theory is strongly coupled at the singularity. The issue of whether this marginal operator is turned on or not in the infrared theory is important. If the operator is turned on there is no weak coupling limit and persumably the infrared behaviour does not look like a perturbative matrix string theory. It is important to resolve this issue further.

5. Conclusions

We have used methods developed for the analysis of infra-red dynamics of $(4,4)$ gauge theories to study the infra-red dynamics of the $(2,2)$ gauge theory on a D1-brane at the conifold. We showed that the infra-red dynamics is captured by a $\mathcal{N} = (2,2)$ superconformal field theory consisting of a linear dilaton with background charge $Q = \sqrt{2}$ and a compact scalar. This agreed with the expectation that the infra-red theory should correspond to that of a fundamental string at the conifold. We mention that these methods can be used to analyze infra-red dynamics of $(2,2)$ theories with one-dimensional Coulomb branch.

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The lagrangian of N D1-branes at the conifold was used to formulate matrix string theory on this background. We note that the leading interaction represented by the twist operator in this case is marginal unlike the case of D1-branes in flat space.

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A. The supergravity solution

We follow the following strategy to verify the supergravity solution in (2.1). We convert the solution given in (2.1) into the Einstein metric and into the conventions of [25]. In these conventions we make the following ansatz for the supergravity solution.

$$
\begin{aligned}
ds^2 &= f^{-3/4}(-dx_0^2 + dx_1^2) + \\
&\quad + f^{1/4}\Bigg[dr_1^2 + r_1^2 d\chi^2 + dr_2^2 + \frac{r_2^2}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 + \\
&\qquad\qquad + \frac{r_2^2}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right)\Bigg], \\
e^{\Phi} &= f^{1/2}, \qquad B_{01} = -f^{-1},
\end{aligned}
$$

where f is an unknown function. We then show that the equations of motion of the various field just reduce to the laplacian for the function f in the coordinates transverse to the D1-brane.

We first substitute this ansatz in the dilaton equation. The dilaton equation in the convention of [25] is given by

$$
\begin{aligned}
&\partial_{MN}\left(\sqrt{-g}g^{MN}\partial_N\Phi\right) - \frac{1}{24}\sqrt{-g}e^{\Phi}H^2 = \\
&\quad = -\frac{\kappa^2 T_2}{2}\int d^2\xi\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M\partial_j X^N g_{MN}e^{-\Phi/2}\delta^{10}(x-X)\,. \qquad \text{(A.1)}
\end{aligned}
$$

Substituting the values of the field given in (A.1) in the static gauge we find that the dilaton equation (A.1) reduces to

$$
\frac{1}{2f}\partial_{r_1}(K\partial_{r_1}f) + \frac{1}{2f}\partial_{r_2}(K\partial_{r_2}f) = -T_2\,\kappa^2 K\frac{108}{(4\pi)^3 f}\frac{\delta(r_1)}{r_1}\frac{\delta(r_2)}{r_2^5}\,, \qquad \text{(A.2)}
$$

where

$$K = \frac{r_1 r_2^5}{108} \sin\theta_1 \sin\theta_2 \,, \tag{A.3}$$

(A.2) is the laplacian for the transverse space. It is clear that the solution of this is given by

$$f = A + \frac{B}{(r_1^2 + r_2^2)^3} \,, \tag{A.4}$$

where A and B are constants.

We now verify that the antisymmetric tensor equation of motion and the Einstein equation also reduces to the laplacian in transverse space (A.2). The antisymmetric tensor equation is

$$\partial_M \left(\sqrt{-g}\, e^{\Phi} H^{MNO}\right) = 2\kappa^2\, T_2 \int d^2\xi \epsilon^{i_1 i_2} \partial_{i_1} X^N \partial_{i_2} X^O \delta^{10}(x - X) \,. \tag{A.5}$$

Substituting the ansatz in (A.1) the equation for the antisymmetric tensor reduces to

$$\partial_{r_1}(K\partial_{r_1} f) + \partial_{r_2}(K\partial_{r_2} f) = -2T_2\kappa^2 K \frac{108}{(4\pi)^3} \frac{\delta(r_1)}{r_1} \frac{\delta(r_2)}{r_2^5} \,, \tag{A.6}$$

(A.2) is identical to the above equation. Therefore the same solution (A.4) satisfies it.

The Einstein equation is given in (3.15) of [25]. After some tedious but straight forward calculations the components of Einstein equation along the D1-brane reduce to

$$\frac{1}{2f^{7/4}} \partial_{r_1}(K\partial_{r_1} f) + \frac{1}{2f^{7/4}} \partial_{r_2}(K\partial_{r_2} f) = -T_2\kappa^2 K \frac{108}{(4\pi)^3 f^{7/4}} \frac{\delta(r_1)}{r_1} \frac{\delta(r_2)}{r_2^5} \,. \tag{A.7}$$

This is the same as (A.2). The remaining components of the Einstein equation reduce to identities for the ansatz in (A.1). The brane field equations are also automatically satisfied.

To fix the constants A and B we go back to the string metric and into the conventions of [4]. $A = 1$ as we require that at infinity the metric reduce to that of the conifold. The constant B by demanding that the net charge of the D1-branes is quantized. This gives that

$$B = N g_s \alpha'^3 \frac{864\pi^2}{16 + 15\pi} \,, \tag{A.8}$$

where N is an integer.

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [`hep-th/9711200`].

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[4] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[5] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025.

[6] T. Banks and N. Seiberg, *Strings from matrices*, *Nucl. Phys.* **B 497** (1997) 41 [hep-th/9702187].

[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[8] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[9] O. Aharony and M. Berkooz, *IR dynamics of $D = 2$, $N = (4,4)$ gauge theories and DLCQ of 'little string theories'*, *J. High Energy Phys.* **10** (1999) 030 [hep-th/9909101].

[10] E. Witten, *On the conformal field theory of the Higgs branch*, *J. High Energy Phys.* **07** (1997) 003 [hep-th/9707093].

[11] M. Berkooz and H. Verlinde, *Matrix theory, AdS/CFT and Higgs-Coulomb equivalence*, *J. High Energy Phys.* **11** (1999) 037 [hep-th/9907100].

[12] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[13] M. Douglas, J. Polchinski and A. Strominger, *Probing five-dimensional black holes with D-branes*, *J. High Energy Phys.* **12** (1997) 003 [hep-th/9703031].

[14] J.S.J. Gates, C.M. Hull and M. Rocek, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157.

[15] D.-E. Diaconescu and N. Seiberg, *The coulomb branch of $(4,4)$ supersymmetric field theories in two dimensions*, *J. High Energy Phys.* **07** (1997) 001 [hep-th/9707158].

[16] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *J. High Energy Phys.* **10** (1999) 035 [hep-th/9907178].

[17] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *J. High Energy Phys.* **10** (1999) 034 [hep-th/9909110].

[18] A. Giveon and D. Kutasov, *Comments on double scaled little string theory*, *J. High Energy Phys.* **01** (2000) 023 [hep-th/9911039].

JHEP05(2000)019

[19] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of $c = 1$ string theory*, *Nucl. Phys.* **B 407** (1993) 667 [hep-th/9301083].

[20] D. Ghoshal and C. Vafa, *$c = 1$ string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[21] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[22] G. Mandal, A.M. Sengupta and S.R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod. Phys. Lett.* **A 6** (1991) 1685.

[23] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.

[24] E. Silverstein and Y.S. Song, *On the critical behavior of D1-brane theories*, *J. High Energy Phys.* **03** (2000) 029 [hep-th/9912244].

[25] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, *Phys. Rep.* **259** (1995) 213 [hep-th/9412184].